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Monty Hall Problem Revisited Once More

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The Monty Hall problem (MHP) is an attractive puzzle. It combines simple statement with answers that seem surprising to most audiences. The problem was thoroughly solved over two decades ago. Yet, more recent discussions indicate that the solution is incompletely understood. Here, we review the solution and discuss pitfalls and other aspects that make the problem interesting.

The MHP, equivalent to the three-prisoner puzzle, is a brain teaser from probability theory. In 1990, vos Savant succinctly presented a solution [1], which was spelled out by the same author in two subsequent notes [2, 3]. Shortly afterwards, her solution was challenged by Gillman [4,5], who argued that she had implicitly considered a modified version of the game, and proceeded to present a thorough solution. Nonetheless, the lucid explanations in Gillman's articles seem not to have been thoroughly absorbed, since more recent discussions have presented incomplete analyses of the problem. Very recent examples, which we single out because they appear in books of outstanding quality, are due to Ben-Naim [6] and Miller [7]. Yet another survey of the various aspects of MHP seems, therefore, warranted.

1 The Monty Hall game

The rules of the game are simple. Monty Hall shows three closed boxes to a contender – call her Portia. One of them contains a car, while a goat stands inside each of the other two. The boxes are symmetrically placed around a circular track, which can be freely rotated around its centre. Portia chooses one of the boxes, and the track is immediately rotated as shown in Figure 1.
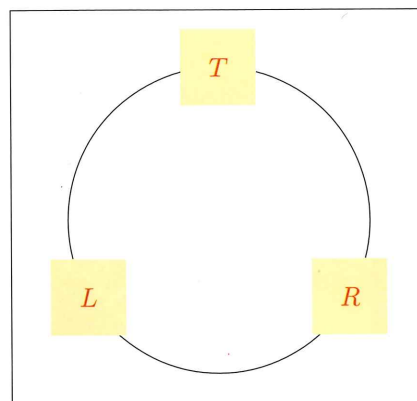

Figure 1: Positions of the three boxes after the contender has stated her initial choice. The structure has been rotated so that the chosen box lies at the top (T). The other two boxes are now at the positions labelled L and R.

Monty Hall knows where the car is. He opens one of the two boxes at the bottom of the figure – box L, for definiteness – to show Portia that it contains a goat, and asks her whether she wants to stick to her choice or switch to box R. The problem asks for the probability P_R that Portia will win the car if she decides to switch.

Monty Hall presents *Let's Make a Deal*, the gameshow that inspired this problem

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2 Gillman's versions

Gillman pointed out that the game admits two variants [4,5]. One of them, which he calls version I, follows the rules in Section 1.

2.1 Gillman's game I

Gillman observes that a complete statement of the game in Section 1 calls for knowledge of Monty Hall's tactics. The host has some freedom. If the car lies in box L (R) in Figure 1, then his only option is to open door R (L). If the car lies in box T, however, the host may open either door. We will call q the probability that he will open door L. To compute her odds, Portia must know q.

Let us assume that she does know q and consider two illustrative situations: $q = 0$ and $q = 1$. With $q = 0$, Monty Hall will blindly open door R whenever given the choice. By opening door L, he in practice admits to Portia that he had no choice, which means that the car must be behind door R. Portia, therefore, asks to switch with the winner's smile on her lips. In this case, the probability of winning by switching is $P_R = 1$.

With $q = 1$, by contrast, if the car is in box T Monty Hall will open box L. If the car is in box R, he will also open box L. Hence, the only piece of information conveyed to Portia when he opens L is the obvious one: that box L holds a goat. To switch or not to switch is then Portia's question, because the chances are equal. In other words, $P_R = 1/2$ for $q = 1$.

We can see that the probability depends on q. To come to the same conclusion from another perspective, consider the probability P_T, of winning without switching. Clearly, $P_T = 1 - P_R$. Before Monty opened the door, the probability was $P_T = 1/3$. When he opens door L, a possibility is eliminated, but this will not change P_T if boxes L and R are equivalent, i.e., if Monty is unbiased towards either box, i.e., if $q = 1/2$. We conclude that, with $q = 1/2$, the probability of winning by not switching is $P_T = 1/3$, and the probability of winning by switching is $P_R = 2/3$.

With $q = 1/2$, therefore, Savant's argument applies, and the probability of winning by switching is $P_R = 2/3$. Clearly, vos Savant had the unbiased game in mind when she wrote on the subject [1–3], i.e., she considered the $q = 1/2$ special case of Gillman's game I. Gillman argued that she had another game in mind. That interpretation seems unfair, even though her solution does apply to Gillman's game II, which we describe next.

2.2 Gillman's game II

In the second version of the game, Portia must announce her decision to switch or not to switch before Monty opens the door. Knowing the host's bias is now of no help to Portia, since she must make her decision before Monty opens either box. In that case, vos Savant's argument is impeccable, for Portia's choice cannot affect the probability $P_T = 1/3$ that the car is in the top box. The probability that Portia wins by switching is, therefore, $2/3$.

3 Formal analysis

We now present mathematical proofs of the results derived on intuitive grounds in Sections 2.1 and 2.2.

3.1 Game I

First solution

We ask for the conditional probability that the car lies in box i ($i = L, R$), given that Monty has opened box j ($j = L, R$). In other words, we ask for the conditional probability $P(C_j \mid H_i)$, where C_j denotes *event j*, i.e., the car is in box j, and H_i denotes event i, i.e., Monty Hall has opened box i. As Portia knows, if the car is in box T, the probability that Monty will open box R is

$$P(H_R \mid C_T) = q. \qquad (1)$$

Let us now turn to Bayes' formula, which states that

$$P(H_R)P(C_L \mid H_R) = P(C_L)P(H_R \mid C_L), \qquad (2)$$

and, likewise, that

$$P(H_R)P(C_T \mid H_R) = P(C_T)P(H_R \mid C_T). \qquad (3)$$

The a priori probabilities that the car is in box L or box T are identical: $P(C_L) = P(C_T) = 1/3$. Division of equation (3) by equation (2), therefore, shows that

$$\frac{P(C_L \mid H_R)}{P(C_T \mid H_R)} = \frac{P(H_R \mid C_L)}{P(H_R \mid C_T)}. \qquad (4)$$

Moreover, $P(H_R \mid C_L) = 1$, because Monty Hall is forced to open box R when the car is in box L. From equations (1) and (4) it follows that

$$\frac{P(C_L \mid H_R)}{P(C_T \mid H_R)} = \frac{1}{q}. \qquad (5)$$

We then recall that events C_T and C_L are mutually exclusive when event H_R takes place (in other words, when Monty opens door R, he is telling Portia that the car is either in box T or in box L). This shows that the numerator and denominator of the fraction on the left-hand side of equation (5) add up to unity:

$$P(C_L \mid H_R) + P(C_T \mid H_R) = 1. \qquad (6)$$

Solution of the system of equations (5) and (6) yields the desired result:

$$P(C_L \mid H_R) = \frac{1}{1 + q}. \qquad (7)$$

As expected from our discussion in Section 2.1, the probability that Portia wins the car by switching, therefore, varies from unity for $q = 0$ to $1/2$ for $q = 1$. When Monty is unbiased, $q = 1/2$ and the probability is $2/3$, as vos Savant predicted.

Second solution

We now follow Isaac [8] to present an alternative solution that seems more attractive because it starts by surveying the space $\mathcal{S}$ of possible events. Let C_i ($i = T, L, R$) denote the event in which the car lies in box i. Let S_k ($k = L, R$) denote the event that Portia switches to box k, let W indicate that Portia wins – she drives the car home – and let L indicate that Portia loses – she trails back home in the company of a goat. Thus, for instance, the event C_R is incompatible with S_L: if the car is behind door R, then Monty is forced to open door L, and Portia would not switch to the open box, inside which she can see a goat.

The sample space comprises four combinations of events:

$$\mathcal{S} = \{(C_T, S_R, L), (C_T, S_L, L), (C_L, S_L, W), (C_R, S_R, W)\}. \tag{8}$$

We can now compute the probability for each combination on the right-hand side of equation (8). Each individual event C_i ($i = T, L, R$) occurs with probability $1/3$. The combined event (C_T, S_R, L) can occur only when, given that the car is in box T, the host chooses to open box L. Its probability is, therefore, $(1/3)(1 - q)$. Likewise, the combination (C_T, S_L, L) will occur only if the host chooses to open box R. Its probability is $(1/3)q$. Finally, the last two combinations, which are independent of Monty's bias, have probability $1/3$ each.

When the host opens door R (event H_R), Portia rules out the first and the last combinations of events on the right-hand side of equation (8). She can then compute her chance of winning by switching from the probabilities for the other two combinations:

$$P_{S_L | H_R} = \frac{\frac{1}{3}}{\frac{q}{3} + \frac{1}{3}}, \tag{9}$$

which yields the desired result

$$P_{S_L | H_R} = \frac{1}{1 + q}, \tag{10}$$

in agreement with equation (7).

3.2 Game II

Gillman's game II is somewhat different. As in game I, Portia chooses a box, which is promptly moved to the top position, as in Figure 1. Monty then announces that he will soon show her a goat inside one of the two boxes at the bottom in Figure 1. Immediately after the announcement, before opening any door, he asks her whether she will switch to the other box. If Portia chooses to switch, what are her chances of winning the car?

First solution
As before, let us call C_i ($i = T, L, R$) the event that the car is in position i. As in Section 3.1, we can see that the probability is $P(C_i) = 1/3$ ($i = T, L, R$). Call $P(W \mid C_i)$ the probability that Portia wins by switching if the car is in box i ($i = T, L, R$). The probability of winning by switching is

$$P(W) = P(W \mid C_T)P(C_T) + \\ P(W \mid C_L)P(C_L) + P(W \mid C_R)P(C_R). \tag{11}$$

If the car is inside box T, switching will make Portia lose. Hence $P(W \mid C_T) = 0$. She knows that, in this case, the probability that Monty will choose box R (box L) is q ($1 - q$), but that information is of no help, because the probability of winning is zero, either way.

On the other hand, $P(W \mid C_L) = 1$, for if the car is in box L, Monty must open box R, and Portia will be compelled to make the right choice, i.e., open box L. Likewise, $P(W \mid C_R) = 1$. It follows from equation (11) that

$$P(W) = 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3}, \tag{12}$$

that is,

$$P(W) = \frac{2}{3}. \tag{13}$$

By contrast, if Portia decides to stick with her initial choice, $P(W \mid C_T)$ will be unitary, while $P(W \mid C_L) = P(W \mid C_R) = 0$. In this case, equation (12) yields $P(W) = 1/3$.

Second solution
As in Section 3.1, let us follow Isaac [8] and consider the four possible combinations of events. The sample space is again given by equation (8), and the probability that each combination occurs is the one we have already computed: $(1 - q)/3$ for (C_T, S_R, L), $q/3$ for (C_T, S_L, L), $1/3$ for (C_L, S_L, W) and $1/3$ for (C_R, S_R, W). In either of the latter two events, Portia drives the car home. The probability of winning by switching is, therefore, the sum of the two probabilities, and we recover equation (13).

Equation (13) agrees with the result derived in [1–3]. This does not mean, however, that vos Savant was playing game II. As already pointed out, all evidence indicates that she was considering game I with $q = 1/2$, i.e., with an unbiased host. Under that condition, equations (7) and (10) are also equivalent to equation (13).

4 A long series of game I

While discussing game I in Section 3.1, we assumed that Monty had opened door R, and our computation of the probability relied on that assumption. If the game is played not once, but numerous times, we cannot assume that box R will be opened every time. In a long series of games, the host's bias becomes irrelevant. The distinction between games I and II is washed out, as the following discussion shows.

In a long sequence, the car will be in box T in $1/3$ of the events. In those events, Monty has his way. Whether he has or has not a preference for one of the doors is entirely immaterial, however, for in the end Portia will take a goat home, if she decides to switch. Of course, if he is biased towards one of the doors, Portia will more often take home the goat in the other box, but that will be of no comfort to her.

In the other $2/3$ of the events, the car will lie either in box R or in box L. In those events, Portia's decision to switch will add a brand new car to her assets. The odds of winning by switching, therefore, are $2/3$, just as if Monty and Portia were playing game II.

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